IOP Publishing Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **50** (2017) 021001 (2pp) doi:10.1088/1751-8121/50/2/021001

Viewpoint

40 years of supersymmetric quantum mechanics[1]

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This viewpoint relates to an article by Sukumar (1985 *J. Phys. A: Math. Gen.* **18** 2917) and was published as part of a series of viewpoints celebrating some of the most influential papers published in *J. Phys.* A, which is celebrating its 50th anniversary.

It was 40 years ago when supersymmetric quantum mechanics was introduced as an algebraic structure in non-relativistic quantum systems. In fact, to the best of the author's knowledge, it was actually the paper by Nicolai [1], published in 1976 in this journal, which explicitly introduced the concept of supersymmetry within the realm of non-relativistic quantum mechanics. Indeed, equation (2.1) of Nicolai's paper already had the basic structure of a supersymmetric quantum system characterised by a Hamiltonian H and the generators Q and $Q^\dagger$ acting on some Hilbert space $\mathcal{H}$ obeying, the algebra

$$\{Q, Q^\dagger\} = H, \qquad Q^2 = 0 = (Q^\dagger)^2. \tag{1}$$

Whereas Nicolai studied supersymmetric spin systems more attention was given to a model pursued by Witten five years later [2]. Witten introduced this as a toy model to study the dynamical breaking of supersymmetric field theories. The Witten model describes a one-dimensional non-relativistic point mass m with an additional spin-$\frac{1}{2}$ degree of freedom. That is, the Hilbert space is given by $\mathcal{H} := L^2(\mathbb{R}) \otimes \mathbb{C}^2$ and the Hamiltonian characterised by a pair of Schrödinger-type Hamilton operators

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \qquad Q^\dagger = \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix}, \tag{2}$$

where $A := \partial_x + \Phi(x)$ with the so-called SUSY potential $\Phi : \mathbb{R} \to \mathbb{R}$ being a continuous differentiable function on the real line. Units are chosen such that $2m = \hbar = 1$. The pair of Hamiltonians explicitly read

$$H_\pm = -\partial_x^2 + \Phi^2(x) \pm \Phi'(x). \tag{3}$$

It was the celebrated paper by Sukumar [3], published 1985 in this journal, which explicitly showed that every one-dimensional Schrödinger Hamiltonian of the standard form $-\partial_x^2 + V(x)$ can be put into a supersymmetric structure thus forming together with its partner Hamiltonian

[1] Dedicated to the memory of Peter Junker.

a supersymmetric quantum system in the above sense. It shall be noted that supersymmetry implies that both partner Hamiltonians are essentially isospectral, that is, they have the same eigenvalues with a possible exception of the ground states energy for one of them. The corresponding eigenstates are related via SUSY transformations. As a consequence the knowledge of the spectral properties of one of the Hamiltonians allows to determine those of the partner Hamiltonian. Similarly for scattering states the reflection and transmission coefficients of the two partner Hamiltonians are identical [4]. As examples, Sukumar considered the free particle and the linear harmonic oscillator systems. In the latter case he was able to construct a family of potentials having identical spectra being related to that of the harmonic oscillator. He also showed the close connections of supersymmetric quantum mechanics with the factorisation method of Schrödinger, Infeld and Hull [5].

Sukumar's idea to utilise supersymmetry for the construction of partners of quantum mechanical systems with known spectral properties has been the subject of numerous studies during the last three decades [6][2]. It has even been extended to so-called PT-symmetric quantum models [7], which have attracted much attention in the last decade [8]. This idea is not limited to non-relativistic quantum mechanics but was also successfully applied, for example, in the construction of new $(1 + 1)$-dimensional field models with classical finite-energy solutions beyond those of the soliton solution of the sine-Gordon equation and the instanton-solution of the ϕ^4-model [9].

Supersymmetry as an algebraic tool is nowadays a well established method in theoretical and mathematical physics. It is not limited to the study of non-relativistic quantum systems but also successfully applied to Pauli- and Dirac-type Hamiltonians as well as to stochastic dynamics described by the Langevin and Fokker–Planck equations. For an overview of such supersymmetric methods in quantum and statistical physics see the monographs [10, 11].

References

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[2] See for example, Junker and Roy [6], which covers basically all factorisable potentials including the harmonic oscillator discussed by Sukumar [3].